

Jawerth-Franke embeddings of Herz-type Besov and Triebel-Lizorkin spaces

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Abstract

In this paper we prove the Jawerth-Franke embeddings of Herz-type Besov and Triebel-Lizorkin spaces. Moreover, we obtain the Jawerth-Franke embeddings of Besov and Triebel-Lizorkin spaces equipped with power weights. An application we present new embeddings between Besov and Herz spaces.

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1 Introduction

The Herz-type Besov-Triebel-Lizorkin spaces initially appeared in the papers of J. Xu and D. Yang [21] and [22]. Several basic properties were established, such as the Fourier analytical characterisation and lifting properties. When $\alpha = 0$ and $p = q$ they coincide with the usual function spaces $F_{p,q}^s$.

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. In [13], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. Also in [18], Y. Tsutsui, studied the Cauchy problem for Navier-Stokes equations on Herz spaces and weak Herz spaces.

The main aim of this paper is to prove the Jawerth-Franke embeddings in $\dot{K}_q^{\alpha,p} F_\beta^s$ and $\dot{K}_q^{\alpha,p} B_\beta^s$ spaces, where we use the so-called φ -transform characterization in the sense of Frazier and Jawerth. As a consequence, we present the Jawerth-Franke embeddings of Besov and Triebel-Lizorkin spaces equipped with power weights. Also, we present new embeddings between Besov and Herz spaces. All these results generalize the existing classical results on Besov and Triebel-Lizorkin spaces.

For any $u > 0, k \in \mathbb{Z}$ we set $C(u) = \{x \in \mathbb{R}^n : u/2 \leq |x| < u\}$ and $C_k = C(2^k)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in \mathbb{R}^n with center x and radius r . Let χ_k , for $k \in \mathbb{Z}$, denote the characteristic function of the set C_k . The expression

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$f \approx g$ means that $Cg \leq f \leq cg$ for some independent constants c, C and non-negative functions f and g .

We denote by $|\Omega|$ the n -dimensional Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$. For any measurable subset $\Omega \subseteq \mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0 < p \leq \infty$ consists of all measurable functions for which $\|f\|_{L^p(\Omega)} = (\int_{\Omega} |f(x)|^p dx)^{1/p} < \infty$, $0 < p < \infty$ and $\|f\|_{L^\infty(\Omega)} = \text{ess-sup}_{x \in \Omega} |f(x)| < \infty$. If $\Omega = \mathbb{R}^n$ we put $L^p(\mathbb{R}^n) = L^p$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on \mathbb{R}^n . We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$. Its inverse is denoted by $\mathcal{F}^{-1}f$. Both \mathcal{F} and \mathcal{F}^{-1} are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

Let \mathbb{Z}^n be the lattice of all points in \mathbb{R}^n with integer-valued components. If $v \in \mathbb{N}_0$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we denote $Q_{v,m}$ the dyadic cube in \mathbb{R}^n ;

$$Q_{v,m} = \{(x_1, \dots, x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, \dots, n\}.$$

By $\chi_{v,m}$ we denote the characteristic function of the cube $Q_{v,m}$.

Given two quasi-Banach spaces X and Y , we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

2 Function spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_q^{\alpha,p}$.

Definition 1 Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_q^p \right)^{1/p},$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p, q) \geq 1$ then $\dot{K}_q^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ then $\dot{K}_p^{0,p}$ coincides with the Lebesgue spaces L^p . A detailed discussion of the properties of these spaces may be found in the papers [9], [11], [12], and references therein.

Now, we present the Fourier analytical definition of Herz-type Besov and Triebel-Lizorkin spaces and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let ϕ_0 be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\phi_0(x) = 1$ for $|x| \leq 1$ and $\phi_0(x) = 0$ for $|x| \geq 2$. We put $\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{1-j}x)$ for $j = 1, 2, 3, \dots$. Then $\{\phi_j\}_{j \in \mathbb{N}_0}$ is a resolution of unity, $\sum_{j=0}^{\infty} \phi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the

Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \mathcal{F}^{-1} \phi_j * f$ of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Besov and Triebel-Lizorkin spaces.

Definition 2 Let $\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty$ and $0 < \beta \leq \infty$.

(i) The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}_q^{\alpha,p} B_\beta^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \phi_j * f\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) Let $0 < p, q < \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p} F_\beta^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}_q^{\alpha,p} F_\beta^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1} \phi_j * f|^\beta \right)^{1/\beta} \right\|_{\dot{K}_q^{\alpha,p}} < \infty, \quad (1)$$

with the obvious modification if $\beta = \infty$.

Remark 1 Let $s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -n/q$. The spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\phi_j\}_{j \in \mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [21], [22], [23] and [25].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

Definition 3 (i) Let $s \in \mathbb{R}$ and $0 < p, \beta \leq \infty$. The Besov space $B_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \phi_j * f\|_p^\beta \right)^{1/\beta} < \infty.$$

(ii) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,\beta}^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1} \phi_j * f|^\beta \right)^{1/\beta} \right\|_p < \infty.$$

The theory of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$ has been developed in detail in [15], [16] and [17] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leq \infty$,

$$\dot{K}_p^{0,p} F_\beta^s = F_{p,\beta}^s.$$

We introduce the sequence spaces associated with the function spaces $\dot{K}_q^{\alpha,p}F_\beta^s$. If

$$\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty$ and $0 < \beta \leq \infty$, we set

$$\|\lambda\|_{\dot{K}_q^{\alpha,p}b_\beta^s} = \left(\sum_{v=0}^{\infty} 2^{vs\beta} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \right\|_{\dot{K}_q^{\alpha,p}}^\beta \right)^{1/\beta}$$

and, with $0 < p, q < \infty$,

$$\|\lambda\|_{\dot{K}_q^{\alpha,p}f_\beta^s} = \left\| \left(\sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{vs\beta} |\lambda_{v,m}|^\beta \chi_{v,m} \right)^{1/\beta} \right\|_{\dot{K}_q^{\alpha,p}}. \quad (2)$$

Let Φ, ψ, φ and Ψ satisfy

$$\Phi, \Psi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \quad (3)$$

$$\text{supp } \mathcal{F}\Phi, \text{supp } \mathcal{F}\Psi \subset \overline{B(0,2)}, \quad |\mathcal{F}\Phi(\xi)|, |\mathcal{F}\Psi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3}, \quad (4)$$

and

$$\text{supp } \mathcal{F}\varphi, \text{supp } \mathcal{F}\psi \subset \overline{B(0,2)} \setminus B(0,1/2), \quad |\mathcal{F}\varphi(\xi)|, |\mathcal{F}\psi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \quad (5)$$

such that

$$\overline{\mathcal{F}\Phi(\xi)} \mathcal{F}\Psi(\xi) + \sum_{j=1}^{\infty} \overline{\mathcal{F}\varphi(2^{-j}\xi)} \mathcal{F}\psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n, \quad (6)$$

where $c > 0$. Recall that the φ -transform S_φ is defined by setting $(S_\varphi f)_{0,m} = \langle f, \Psi_m \rangle$ where $\Psi_m(x) = \Psi(x - m)$ and $(S_\varphi f)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{vn/2} \varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse φ -transform T_ψ is defined by

$$T_\psi \lambda = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m},$$

where $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, see [5].

For simplicity, in what follows, we use $\dot{K}_p^{\alpha,q}A_\beta^s$ to denote either $\dot{K}_p^{\alpha,q}F_\beta^s$ or $\dot{K}_p^{\alpha,q}B_\beta^s$. If $\dot{K}_p^{\alpha,q}A_\beta^s$ means $\dot{K}_p^{\alpha,q}F_\beta^s$ then the case $p = \infty$ is excluded. To prove the main results of this paper we need the following theorem, see [3].

Theorem 1 *Let $\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -n/q$. Suppose that φ and Φ satisfy (3)-(6). The operators $S_\varphi : \dot{K}_q^{\alpha,p}A_\beta^s \rightarrow \dot{K}_q^{\alpha,p}A_\beta^s$ and $T_\psi : \dot{K}_q^{\alpha,p}A_\beta^s \rightarrow \dot{K}_q^{\alpha,p}A_\beta^s$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\dot{K}_q^{\alpha,p}A_\beta^s$.*

We end this section with one more lemma, which is basically a consequence of Hardy's inequality in the sequence Lebesgue space ℓ_q .

Lemma 1 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=0}^k a^{k-j} \varepsilon_j$ and $\eta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j, k \in \mathbb{N}_0$. Then there exists constant $c > 0$ depending only on a and q such that*

$$\left(\sum_{k=0}^{\infty} \delta_k^q \right)^{1/q} + \left(\sum_{k=0}^{\infty} \eta_k^q \right)^{1/q} \leq c \left(\sum_{k=0}^{\infty} \varepsilon_k^q \right)^{1/q}.$$

3 Jawerth embedding

The classical Jawerth embedding says that:

$$F_{q,\infty}^{s_2} \hookrightarrow B_{s,q}^{s_1}$$

if $s_1 - n/s = s_2 - n/q$ and $0 < q < s < \infty$, see e.g. [7]. We will extend this embeddings to Herz-type Besov Triebel-Lizorkin spaces. We follow some ideas of Vybíral, [19], where use the technique of non-increasing rearrangement.

Definition 4 *Let μ be the Lebesgue measure in \mathbb{R}^n . If f is a measurable function on \mathbb{R}^n , we define the non-increasing rearrangement of f through*

$$f^*(t) = \sup\{\lambda > 0 : m_f(\lambda) > t\}$$

where m_f is the distribution function of f .

We shall use the following properties. If $0 < p < \infty$, then

$$\|f\|_p = \|f^* \mid L^p(0, \infty)\| \quad (7)$$

for every measurable function f . Let f and g be two non-negative measurable functions on \mathbb{R}^n . If $1 \leq p \leq \infty$, then

$$\|f + g\|_p \leq \|f^* + g^* \mid L^p(0, \infty)\|. \quad (8)$$

The proof follows from Theorems 3.4 and 4.6 in [1]. First, we will prove the discrete version of Jawerth embedding.

Theorem 2 *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p \leq \infty, 0 < q, r < \infty, \alpha_1 > -n/s$ and $\alpha_2 > -n/q$. We suppose that*

$$s_1 - n/s - \alpha_1 = s_2 - n/q - \alpha_2. \quad (9)$$

Under the following assumptions

$$0 < q < s \leq \infty, q \leq r \text{ and } \alpha_2 > \alpha_1 \quad (10)$$

or

$$0 < q < \min(s, p), q \leq r \leq \min(s, p) \text{ and } \alpha_2 = \alpha_1 \quad (11)$$

or

$$0 < s \leq q < \infty, \alpha_2 + n/q > \alpha_1 + n/s \quad (12)$$

or

$$0 < s \leq q < \infty, q \leq r \leq p \leq \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s,$$

we have

$$\dot{K}_q^{\alpha_2, r} f_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1, p} b_r^{s_1}, \quad (13)$$

where

$$\theta = \begin{cases} r & \text{if } 0 < s \leq q < \infty, q \leq r \leq p \leq \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in \dot{K}_q^{\alpha_2, r} f_\theta^{s_2}$. We have

$$\begin{aligned} \|\lambda\|_{\dot{K}_s^{\alpha_1, p} b_r^{s_1}}^r &= \sum_{v=0}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{(k\alpha_1 + vs_1)p} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^p \right)^{r/p} \\ &\leq \sum_{v=0}^{\infty} \left(\sum_{k=-\infty}^{-v} \dots \right)^{r/p} + \sum_{v=0}^{\infty} \left(\sum_{k=-v}^{\infty} \dots \right)^{r/p} \\ &= I + II. \end{aligned}$$

Step 1. We prove our embedding under the assumption (10) and we will estimate I and II , respectively. We will treat the case only where $0 < s, p < \infty$. The case $s = \infty$ follows by the embedding

$$\dot{K}_{p_0}^{\alpha_1, p} b_r^{s_1 + \frac{n}{p_0} - \frac{n}{s}} \hookrightarrow \dot{K}_s^{\alpha_1, p} b_r^{s_1} \quad (14)$$

for some $0 < q < p_0 < s \leq \infty$, see [2, Theorem 5.9].

Estimation of I . Let $x \in C_k \cap Q_{v,m}$ and $y \in Q_{v,m}$. We have $|x - y| \leq 2\sqrt{n}2^{-v} < 2^{c_n - v}$ and from this it follows that $|y| < 2^{c_n - v} + 2^k \leq 2^{c_n - v + 2}$, which implies that y is located in some ball $B(0, 2^{c_n - v + 2})$ and

$$|\lambda_{v,m}|^t \lesssim 2^{nv} \int_{B(0, 2^{c_n - v + 2})} |\lambda_{v,m}|^t \chi_{v,m}(y) dy,$$

where $t > 0$. Then for any $x \in C_k$ we obtain

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(x) &\lesssim 2^{nv} \int_{B(0, 2^{c_n - v + 2})} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(y) dy \\ &= 2^{nv} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t^t. \end{aligned}$$

Consequently,

$$2^{k\alpha_1 + vs_1} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s \lesssim 2^{v(s_1 + \frac{n}{t} + k(\alpha_1 + \frac{n}{s}))} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t.$$

We may choose $t > 0$ such that $\frac{1}{t} > \max(\frac{1}{q}, \frac{1}{q} + \frac{\alpha_2}{n})$. Therefore, since $\alpha_1 + \frac{n}{s} > 0$,

$$I \lesssim \sum_{v=0}^{\infty} 2^{v(s_1 + \frac{n}{t} - \alpha_1 - \frac{n}{s} - s_2)r} \sup_{j \geq 0} 2^{js_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \chi_{j,m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t^r,$$

which can be estimated by, using (9),

$$c \sum_{v=0}^{\infty} 2^{v \frac{nr}{d}} \left(\sum_{i=-\infty}^{-v} 2^{i \frac{n\sigma}{d} + \alpha_2 \sigma i} \sup_{j \geq 0} 2^{js_2 \sigma} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \chi_{j,m} \chi_{i+c_n+2} \right\|_q^\sigma \right)^{r/\sigma},$$

by Hölder's inequality, with $\sigma = \min(1, t)$ and $\frac{n}{d} = \frac{n}{t} - \frac{n}{q} - \alpha_2$. Hence Lemma 1 implies that

$$I \lesssim \sum_{i=0}^{\infty} 2^{-\alpha_2 i r} \sup_{j \geq 0} 2^{js_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \chi_{j,m} \chi_{2^{-i} + c_n} \right\|_q^r \lesssim \|\lambda\|_{\dot{K}_q^{\alpha_2, r} f_\theta^{s_2}}^r.$$

Estimation of II. Our estimate use partially some decomposition techniques already used in [19]. Set

$$h_k(x) = \sup_{v \geq 0} 2^{vs_2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(x) \chi_k(x).$$

Then

$$\|\lambda\|_{\dot{K}_q^{\alpha_2, r} f_\infty^{s_2}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r} \|h_k\|_q^r \right)^{1/r}$$

and

$$|\lambda_{v,m}| \leq 2^{-vs_2} \inf_{x \in Q_{v,m}} h_k(x), \quad v \in \mathbb{N}_0, m \in \mathbb{Z}^n.$$

Using the fact that $\alpha_2 > \alpha_1$ and the assumption (9) we estimate II by

$$\begin{aligned} & \sum_{v=0}^{\infty} 2^{v(\frac{n}{s} - \frac{n}{q} + s_2)r} \sup_{k \geq -v} 2^{k\alpha_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^r \\ & \leq \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r} \left(\sum_{v=0}^{\infty} 2^{v(\frac{n}{s} - \frac{n}{q} + s_2)rt} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^{rt} \right)^{1/t}, \end{aligned} \quad (15)$$

where

$$\frac{q}{r} < t < \min(1, \frac{s}{r}) \text{ if } q < r \text{ and } t = 1 \text{ if } q = r.$$

We can easily prove the estimate:

$$2^{vs_2 s} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^s \leq 2^{-vn} \sum_{m \in \mathbb{Z}^n} \left(\inf_{x \in Q_{v,m}} h_k(x) \right)^s.$$

Therefore, the sum $\sum_{v=0}^{\infty} \cdots$ in (15) can be estimated by

$$\begin{aligned} & \sum_{v=0}^{\infty} 2^{-\frac{vnrt}{q}} \left(\sum_{m \in \mathbb{Z}^n} \left(\inf_{x \in Q_{v,m}} h_k(x) \right)^s \right)^{rt/s} \\ & \leq \sum_{v=0}^{\infty} 2^{-\frac{vnrt}{q}} \left(\sum_{i=1}^{\infty} ((h_k)^* (2^{-vn} i))^s \right)^{rt/s}. \end{aligned}$$

Using the monotonicity of h and the inequality $rt < s$, the last term is bounded by

$$\begin{aligned} & c \sum_{v=0}^{\infty} 2^{-\frac{vnrt}{q}} \left(\sum_{l=0}^{\infty} 2^{nl} ((h_k)^* (2^{(l-v)n}))^s \right)^{rt/s} \\ & \lesssim \sum_{v=0}^{\infty} 2^{-\frac{vnrt}{q}} \sum_{l=0}^{\infty} 2^{nl \frac{rt}{s}} ((h_k)^* (2^{(l-v)n}))^{rt} \\ & = c \sum_{v=0}^{\infty} 2^{-\frac{vnrt}{q}} \sum_{j=-v}^{\infty} 2^{n(j+v) \frac{rt}{s}} ((h_k)^* (2^{nj}))^{rt} \\ & = c \sum_{j=-\infty}^{\infty} 2^{\frac{jnrt}{s}} ((h_k)^* (2^{nj}))^r \sum_{v=-j}^{\infty} 2^{nv(\frac{1}{s} - \frac{1}{q})rt} \\ & = c \sum_{j=-\infty}^{\infty} 2^{\frac{jnrt}{q}} ((h_k)^* (2^{nj}))^{rt} \end{aligned}$$

Since $q < rt$, using the embedding $\ell_q \hookrightarrow \ell_{rt}$, we get

$$\sum_{j=-\infty}^{\infty} 2^{\frac{n j r t}{q}} ((h_k)^* (2^{n j}))^{rt} \leq \left(\sum_{j=-\infty}^{\infty} 2^{n j} ((h_k)^* (2^{n j}))^q \right)^{rt/q} = \|h_k\|_q^{rt}.$$

Consequently, we obtain $II \lesssim \|\lambda\|_{\dot{K}_q^{\alpha_2, r} f_\infty^{s_2}}$.

Step 2. We prove our embedding under the assumption (11) and we need only to estimate II . Since $q \leq r \leq \min(s, p)$, we can estimate II by (15) with 1 in place of t . Using similar arguments of Step 1, we get the desired estimate. Notice that the case $s = \infty$ follows by the embedding (14) for some $0 < q < p_0 < \min(s, p) \leq \infty$.

Step 3. We prove our embedding under the assumption (12) and again we need only to estimate II . By Hölder's inequality we get

$$2^{vs_1} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \chi_k \right\|_s \leq 2^{(\frac{n}{s} - \frac{n}{q})k + vs_1} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_q.$$

Hence II can be estimated by

$$\begin{aligned} & c \sum_{v=0}^{\infty} 2^{vs_1 r} \left(\sum_{k=-v}^{\infty} 2^{k(\alpha_1 + \frac{n}{s} - \frac{n}{q})p} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_q^p \right)^{r/p} \\ & \leq \sum_{v=0}^{\infty} 2^{vs_2 r} \left(\sum_{k=-v}^{\infty} 2^{(k+v)(\alpha_1 - \alpha_2 + \frac{n}{s} - \frac{n}{q})p} 2^{k\alpha_2 p} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_q^p \right)^{r/p} \\ & \leq \sum_{v=0}^{\infty} \left(\sum_{k=-v}^{\infty} 2^{(k+v)(\alpha_1 - \alpha_2 + \frac{n}{s} - \frac{n}{q})p} 2^{k\alpha_2 p} \left\| \sup_{j \geq 0} 2^{js_2} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \chi_{j,m} \chi_k \right\|_q^p \right)^{r/p} \\ & \lesssim \sum_{v=-\infty}^{\infty} 2^{k\alpha_2 r} \left\| \sup_{j \geq 0} 2^{js_2} \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}| \chi_{j,m} \chi_k \right\|_q^r \\ & \lesssim \|\lambda\|_{\dot{K}_q^{\alpha_2, r} f_\infty^{s_2}}^r, \end{aligned}$$

by Lemma 1. If $\alpha_2 + n/q = \alpha_1 + n/s$ and $r \leq p$, then

$$\begin{aligned} II & \lesssim \sum_{v=0}^{\infty} 2^{vs_2 r} \sum_{k=-v}^{\infty} 2^{k\alpha_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_q^r \\ & \lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r} \left\| \left(\sum_{v=0}^{\infty} 2^{vs_2 r} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}|^r \chi_{v,m} \chi_k \right)^{1/r} \right\|_q^r \\ & \leq \|\lambda\|_{\dot{K}_q^{\alpha_2, r} f_r^{s_2}}^r. \end{aligned}$$

The proof is complete. \blacksquare

We would like to mention that r on the right hand side of (13) is optimal. Indeed, for $v \in \mathbb{N}_0$ and $N \geq 1$, we put

$$\lambda_{v,m}^N = \begin{cases} 2^{-(s_1 - \frac{1}{s} - \alpha_1)v} \sum_{i=1}^N \chi_i(2^{v-1}) & \text{if } m = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\lambda^N = \{\lambda_{v,m}^N : v \in \mathbb{N}_0, m \in \mathbb{Z}\}$. We have

$$\|\lambda^N\|_{\dot{K}_q^{\alpha_2, r} f_\theta^{s_2}}^r = \sum_{k=-\infty}^{\infty} 2^{\alpha_2 k r} \left\| \left(\sum_{v=0}^{\infty} 2^{vs_2 \theta} |\lambda_{v,1}^N|^\theta \chi_{v,1} \right)^{1/\theta} \chi_k \right\|_q^r.$$

We can rewrite the last statement as follows:

$$\begin{aligned}
& \sum_{k=1-N}^0 2^{\alpha_2 k r} \left\| \left(\sum_{v=1}^N 2^{(s_2-s_1+\frac{1}{s}+\alpha_1)v\theta} \chi_{v,1} \right)^{1/\theta} \chi_k \right\|_q^r \\
&= \sum_{k=1-N}^0 2^{\alpha_2 k r} \left\| 2^{(s_2-s_1+\frac{1}{s}+\alpha_1)(1-k)} \chi_{1-k,1} \right\|_q^r \\
&= c N,
\end{aligned}$$

where the constant $c > 0$ does not depend on N . Now

$$\|\lambda^N\|_{\dot{K}_s^{\alpha_1,p} b_{\sigma^1}^{s_1}}^\sigma = \sum_{v=0}^\infty 2^{vs_1\sigma} \left(\sum_{k=-\infty}^\infty 2^{\alpha_1 k p} \left\| \sum_{m \in \mathbb{Z}} |\lambda_{v,m}^N| \chi_{v,m} \chi_k \right\|_s^p \right)^{\sigma/p}.$$

Again we can rewrite the last statement as follows:

$$\|\lambda^N\|_{\dot{K}_s^{\alpha_1,p} b_{\sigma^1}^{s_1}}^\sigma = \sum_{v=1}^N 2^{(\frac{1}{s}+\alpha_1)v\sigma} \left(\sum_{k=1-N}^0 2^{\alpha_1 k p} \left\| \chi_{v,1} \chi_k \right\|_s^p \right)^{\sigma/p} = cN,$$

where the constant $c > 0$ does not depend on N . If the embeddings (13) holds then for any $N \in \mathbb{N}$, $N^{\frac{1}{\sigma}-\frac{1}{r}} \leq C$. Thus, we conclude that $0 < r \leq \sigma < \infty$ must necessarily hold by letting $N \rightarrow +\infty$.

Using Theorems 1 and 2, we have the following Jawerth embedding.

Theorem 3 *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p < \infty$, $0 < q, r \leq \infty$, $\alpha_1 > -n/s$ and $\alpha_2 > -n/q$. Under the hypothesis of Theorem 2 we have*

$$\dot{K}_q^{\alpha_2,r} F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} B_r^{s_1}, \quad (16)$$

where

$$\theta = \begin{cases} r & \text{if } 0 < s \leq q < \infty, q \leq r \leq p \leq \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s \\ \infty & \text{otherwise.} \end{cases}$$

From this theorem and the fact that $\dot{K}_q^{0,q} F_\infty^{s_2} = F_{q,\infty}^{s_2}$ and $\dot{K}_s^{0,p} B_q^{s_1} \hookrightarrow \dot{K}_s^{0,s} B_q^{s_1} = B_{s,q}^{s_2}$, with $0 < p < s < \infty$, we obtain the following embeddings

$$F_{q,\infty}^{s_2} \hookrightarrow \dot{K}_s^{0,p} B_q^{s_1} \hookrightarrow B_{s,q}^{s_2},$$

if

$$0 < q < p < s < \infty \text{ and } s_1 - n/s = s_2 - n/q.$$

From this theorem and the fact that $\dot{K}_q^{\alpha,r} F_2^0 = \dot{K}_q^{\alpha,r}$ for $1 < r, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$, see [20] and again $\dot{K}_s^{0,s} B_r^{s_1} = B_{s,r}^{s_1}$ we immediately arrive at the following embedding between Herz and Besov spaces.

Theorem 4 *Let $\alpha, s_1 \in \mathbb{R}$, $0 < s \leq \infty$, $1 < r, q < \infty$ and $0 \leq \alpha < n - \frac{n}{q}$. We suppose that $\frac{n}{s} - s_1 = \alpha + \frac{n}{q}$. Let*

$$0 < q < s \leq \infty, q \leq r \text{ and } \alpha > 0$$

or

$$0 < q < s \leq \infty, q \leq r \leq s \text{ and } \alpha = 0$$

or

$$0 < s \leq q < \infty, \alpha > \frac{n}{s} - \frac{n}{q}$$

or

$$0 < s \leq q \leq \infty, q \leq r \leq s \leq \infty \text{ and } \alpha = \frac{n}{s} - \frac{n}{q}.$$

Then

$$\dot{K}_q^{\alpha,r} \hookrightarrow B_{s,r}^{s_1},$$

where

$$r = 2 \text{ if } 0 < s \leq q < \infty, q \leq 2 \leq s \leq \infty \text{ and } \alpha = \frac{n}{s} - \frac{n}{q}.$$

Some embeddings between Herz spaces and homogenous Besov spaces can be found in [18].

4 Franke embedding

The classical Franke embedding may be rewritten as follows:

$$B_{q,s}^{s_2} \hookrightarrow F_{s,\infty}^{s_1},$$

if $s_1 - n/s = s_2 - n/q$ and $0 < q < s < \infty$, see e.g. [4]. As in Section 3 we will extend this embeddings to Herz-type Besov-Triebel-Lizorkin spaces. Again, we follow some ideas of Vybíral, [19]. We will prove the discrete version of Franke embedding.

Theorem 5 *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, p, q < \infty$, $0 < \theta \leq \infty$, $\alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. We suppose that*

$$s_1 - \frac{n}{s} - \alpha_1 = s_2 - \frac{n}{q} - \alpha_2.$$

Let

$$0 < q < s < \infty, \alpha_2 \geq \alpha_1, \tag{17}$$

or

$$0 < s \leq q < \infty \text{ and } \alpha_2 + \frac{n}{q} > \alpha_1 + \frac{n}{s}. \tag{18}$$

Then

$$\dot{K}_q^{\alpha_2,p} b_p^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} f_\theta^{s_1}. \tag{19}$$

Proof. We prove our embedding under the conditions (17). Let $\lambda \in \dot{K}_q^{\alpha_2,p} b_p^{s_2}$. We have

$$\begin{aligned} \|\lambda\|_{\dot{K}_s^{\alpha_1,p} f_\theta^{s_1}}^p &= \sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p} \left\| \left(\sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v s_1 \theta} |\lambda_{v,m}|^\theta \chi_{v,m} \chi_k \right)^{1/\theta} \right\|_s^p \\ &= \sum_{k=-\infty}^0 \cdots + \sum_{k=1}^{\infty} \cdots \\ &= J_1 + J_2. \end{aligned}$$

Estimation of J_1 . Let $c_n = 1 + [\log_2(2\sqrt{n} + 1)]$. Obviously

$$\begin{aligned} J_1 &\lesssim \sum_{k=-\infty}^0 2^{k\alpha_1 p} \left\| \left(\sum_{v=0}^{c_n-k+1} \dots \right)^{1/\theta} \right\|_s^p + \sum_{k=-\infty}^0 2^{k\alpha_1 p} \left\| \left(\sum_{v=c_n-k+2}^{\infty} \dots \right)^{1/\theta} \right\|_s^p \\ &= T_1 + T_2. \end{aligned}$$

The same analysis as in the proof of Theorem 2 shows that

$$\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(x) \lesssim 2^{nv} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \chi_{B(0, 2^{c_n-v+2})} \right\|_t^t$$

for any $x \in C_k$. From Lemma 1, since $\alpha_1 + \frac{n}{s} > 0$, T_1 does not exceed

$$c \sum_{v=0}^{\infty} 2^{v(s_1 - \alpha_1 - \frac{n}{s} + \frac{n}{t})p} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \chi_{B(0, 2^{c_n-v+2})} \right\|_t^p.$$

We may choose $t > 0$ such that $\frac{1}{t} > \max(\frac{1}{q}, \frac{1}{q} + \frac{\alpha_2}{n})$, $\sigma = \min(1, t)$ and $\frac{n}{d} = \frac{n}{t} - \frac{n}{q} - \alpha_2$. By Hölder's inequality, this term is bounded by

$$c \sum_{v=0}^{\infty} 2^{v \frac{n}{d} p} \left(\sum_{i=-v}^{\infty} 2^{i \frac{n}{d} + \alpha_2 \sigma i} \sup_{j \geq 0} \left\| \sum_{m \in \mathbb{Z}^n} 2^{s_2 j} |\lambda_{j,m}| \chi_{j,m} \chi_{i+c_n+2} \right\|_q^{\sigma} \right)^{p/\sigma}.$$

Using again Lemma 1, the last term is bounded by

$$c \sum_{i=0}^{\infty} 2^{-\alpha_2 i p} \sup_{j \geq 0} \left\| \sum_{m \in \mathbb{Z}^n} 2^{s_2 j} |\lambda_{j,m}| \chi_{j,m} \chi_{2^{-i}+c_n} \right\|_q^p \lesssim \|\lambda\|_{\dot{K}_q^{\alpha_2, p} b_p^{s_2}}^p.$$

Now we estimate T_2 . First let us consider $\alpha_2 > \alpha_1$. We have

$$\begin{aligned} T_2 &\lesssim \sum_{k=-\infty}^0 2^{k\alpha_2 p} \sup_{v \geq c_n+2-k} 2^{v(s_2 - \frac{n}{q} + \frac{n}{s})p} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^p \\ &= \sum_{k=-\infty}^0 2^{k\alpha_2 p} \sup_{v \geq c_n+2-k} 2^{v(s_2 - \frac{n}{q} + \frac{n}{s})p} \|h_{v,k}\|_s^p. \end{aligned}$$

Let us prove that

$$2^{v(\frac{n}{s} - \frac{n}{q})} \|h_{v,k}\|_s \lesssim \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_{\tilde{C}_k} \right\|_q = \delta,$$

where $\tilde{C}_k = \cup_{i=-1}^2 C_{k+i}$, wich equivalent to

$$\int 2^{v(\frac{n}{s} - \frac{n}{q})s} (h_{v,k}(x) \delta^{-1})^s dx \lesssim 1.$$

Let $x \in C_k \cap Q_{v,m}$ and $y \in Q_{v,m}$ with $v \geq c_n - k + 2$. We have $|x - y| \leq 2\sqrt{n}2^{-v} < 2^{c_n-v}$ and from this it follows that $2^{k-2} < |y| < 2^{c_n-v} + 2^k < 2^{k+2}$, which implies that y is located in \tilde{C}_k . Therefore,

$$|\lambda_{v,m}|^q \chi_{v,m}(x) \lesssim 2^{nv} \int_{\tilde{C}_k} |\lambda_{v,m}|^q \chi_{v,m}(y) dy,$$

if $x \in C_k \cap Q_{v,m}$. We have

$$\begin{aligned} \int 2^{v(\frac{n}{s}-\frac{n}{q})s} (h_{v,k}(x)\delta^{-1})^s dx &= \int \left(2^{-v\frac{n}{q}} h_{v,k}(x)\delta^{-1}\right)^{s-q} (h_{v,k}(x)\delta^{-1})^q dx \\ &\lesssim \int (h_{v,k}(x)\delta^{-1})^q dx \\ &\lesssim 1. \end{aligned}$$

Consequently,

$$T_2 \lesssim \sum_{k=-\infty}^0 2^{k\alpha_2 p} \sup_{v \geq 0} 2^{vs_2 p} \|h_{v,k}\|_q^p \leq \|\lambda\|_{\dot{K}_q^{\alpha_2, p, b_p^{s_2}}}^p.$$

Now let us consider $\alpha_2 = \alpha_1$. We can suppose that $\theta \leq q$ and $p \leq s$, since the opposite cases can be obtained by the fact that $\ell_q \hookrightarrow \ell_\theta$ and/or $\ell_s \hookrightarrow \ell_p$, respectively. Observe that

$$|2^{-v}m| \leq |x - 2^{-v}m| + |x| \leq 2^k + \sqrt{n}2^{-v} \leq 2^{k+1}$$

and

$$|2^{-v}m| \geq ||x - 2^{-v}m| - |x|| \geq 2^{k-1} - \sqrt{n}2^{-v} \geq 2^{k-2}$$

if $x \in C_k \cap Q_{v,m}$ and $v \geq c_n + 2 - k$. Hence m is located in

$$\bar{C}_{k+v} = \{m \in \mathbb{Z}^n : 2^{k+v-2} \leq |m| \leq 2^{k+v+1}\}.$$

Therefore T_2 can be estimated by

$$\sum_{k=-\infty}^0 2^{k\alpha_1 p} \left\| \left(\sum_{v=c_n+2-k}^{\infty} \sum_{m \in \bar{C}_{k+v}} 2^{vs_1 \theta} |\lambda_{v,m}|^\theta \chi_{v,m} \right)^{1/\theta} \right\|_s^p.$$

Let

$$\tilde{\lambda}_{v,m_1}^{1,k} = \max_{m \in \bar{C}_{k+v}} |\lambda_{v,m}|, \quad m_1 \in \mathbb{Z}^n$$

and (decreasing rearrangement of $\{\lambda_{v,m}\}_{m \in \bar{C}_{k+v}}$)

$$\tilde{\lambda}_{v,m_j}^{j,k} = \max_{m^i \in \bar{C}_{k+v}, i=1, \dots, j} \sum_{i=1}^j |\lambda_{v,m^i}| - \sum_{i=1}^{j-1} \tilde{\lambda}_{v,m_i}^{i,k}, \quad m_j \in \mathbb{Z}^n, j \geq 2.$$

Then

$$\sum_{m \in \bar{C}_{k+v}} |\lambda_{v,m}| \chi_{v,m} = \sum_{i=1}^{L_{k+v}} \tilde{\lambda}_{v,m_i}^{i,k} \chi_{v,m_i} = f$$

for some $L_{k+v} \in \mathbb{N}$. It is not difficult to see that

$$f^*(t) = \sum_{i=1}^{L_{k+v}} |\tilde{\lambda}_{v,m_i}^{i,k}| \tilde{\chi}_{[B_{i-1}, B_i)}(t),$$

with

$$B_i = \sum_{j=1}^i |Q_{v,m_j}| = 2^{-vn} i, \quad i = 1, \dots, L_{k+v}.$$

Using the properties (7) and (8) we can estimate $\|\cdots\|_s^p$ by

$$c \left\| \sum_{v=c_n+2-k}^{\infty} \sum_{i=1}^{L_{k+v}} 2^{vs_1\theta} |\tilde{\chi}_{v,m_i}^{i,k}|^\theta \tilde{\chi}_{v,i} \mid L^{s/\theta}(0, \infty) \right\|^{p/\theta},$$

where $\tilde{\chi}_{v,i}$ is a characteristic function of the interval $(2^{-v}(i-1), 2^{-v}i)$. By duality, the last norm may be rewritten as

$$\sup \int_0^\infty \sum_{v=c_n+2-k}^{\infty} \sum_{i=1}^{L_{k+v}} 2^{vs_1\theta} |\tilde{\chi}_{v,m_i}^{i,k}|^\theta \tilde{\chi}_{v,i}(x) g(x) dx \quad (20)$$

where the supremum is taken over all non-increasing non-negative measurable functions g with $\|g \mid L^\beta(0, \infty)\| \leq 1$ and β is the conjugated index to s/θ . Similarly, ϱ stands for the conjugated index to q/θ . Let

$$g_{v,i} = \int_0^\infty \tilde{\chi}_{v,i}(x) g(x) dx.$$

Hölder's inequality implies that

$$\begin{aligned} & \sum_{v=c_n+2-k}^{\infty} \sum_{i=1}^{L_{k+v}} 2^{vs_1\theta} |\tilde{\chi}_{v,m_i}^{i,k}|^\theta \tilde{\chi}_{v,i} g_{v,i} \\ & \leq \sum_{v=c_n+2-k}^{\infty} \left(\sum_{i=1}^{L_{k+v}} 2^{vs_2q} |\tilde{\chi}_{v,m_i}^{i,k}|^q \right)^{\theta/q} \left(\sum_{h=1}^{\infty} 2^{v(s_1-s_2)\theta\varrho} g_{v,h}^\varrho \right)^{1/\varrho} \\ & \leq \left(\sum_{v=c_n+2-k}^{\infty} \left(\sum_{i=1}^{L_{k+v}} 2^{v(s_2-\frac{n}{q})q} |\tilde{\chi}_{v,m_i}^{i,k}|^q \right)^{s/q} \right)^{\theta/s} \left(\sum_{v=c_n+2-k}^{\infty} \left(\sum_{h=1}^{\infty} 2^{\frac{vn\theta\varrho}{s}} g_{v,h}^\varrho \right)^{\beta/\varrho} \right)^{1/\beta}. \end{aligned}$$

As in [19] we can prove that the second term is bounded. Clearly the first term can be estimated by

$$c \left(\sum_{v=c_n+2-k}^{\infty} \left(\sum_{m \in \bar{C}_{k+v}} 2^{v(s_2-\frac{n}{q})q} |\lambda_{v,m}|^q \right)^{s/q} \right)^{\theta/s} \lesssim \left(\sum_{v=1}^{\infty} 2^{vs_2s} \left\| \sum_{m \in \bar{C}_{k+v}} \lambda_{v,m} \chi_{v,m} \check{C}_k \right\|_q^s \right)^{\theta/s},$$

where $\check{C}_k = \cup_{i=-2}^3 C_{k+i}$. Using the well-known inequality

$$\left(\sum_{j=0}^{\infty} |a_j| \right)^\rho \leq \sum_{j=0}^{\infty} |a_j|^\rho, \quad \{a_j\}_j \subset \mathbb{C}, \quad \rho \in (0, 1], \quad (21)$$

we obtain that T_2 can be estimated by $c \|\lambda\|_{\dot{K}_q^{\alpha_2, p} b_p^{s_2}}^p$.

Estimation of J_2 . We use the same notations as in the estimation of J_1 . We have

$$J_2 \leq \sum_{k=1}^{c_n+1} \cdots + \sum_{k=c_n+2}^{\infty} \cdots.$$

As in the estimation of T_2 , the second term can be estimated by $c \|\lambda\|_{\dot{K}_q^{\alpha_2, p} b_p^{s_2}}^p$. Now the first term is bounded by

$$\begin{aligned} & c \sum_{k=1}^{c_n+1} 2^{k\alpha_1 p} \left\| \left(\sum_{v=0}^{c_n-k+1} \dots \right)^{1/\theta} \right\|_s^p + \sum_{k=1}^{c_n+1} 2^{k\alpha_1 p} \left\| \left(\sum_{v=c_n-k+2}^{\infty} \dots \right)^{1/\theta} \right\|_s^p \\ & \lesssim \|\lambda\|_{\dot{K}_q^{\alpha_2, p} b_p^{s_2}}^p, \end{aligned}$$

where again we used the same arguments as in the estimation of T_1 and T_2 .

Using a combination of the arguments used in Step 3 of the proof Theorem 2 we prove our embedding under the conditions (18). The proof is complete. \blacksquare

Also as above p on the right hand side of (19) is optimal.

Using Theorems 1 and 5, we have the following Franke embedding.

Theorem 6 *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, q < \infty$, $0 < p < \infty$, $0 < \theta \leq \infty$, $\alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. Under the hypothesis of Theorem 5 we have*

$$\dot{K}_q^{\alpha_2, p} B_p^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1, p} F_{\theta}^{s_1}. \quad (22)$$

We would like to mention that from this theorem we have

$$B_{q, s}^{s_2} \hookrightarrow \dot{K}_q^{0, s} B_s^{s_2} \hookrightarrow F_{s, \theta}^{s_1},$$

if $0 < q < s < \infty$, $0 < \theta \leq \infty$ and

$$s_1 - \frac{n}{s} = s_2 - \frac{n}{q}.$$

Also we immediately arrive at the following embedding between Herz and Besov spaces.

Theorem 7 *Let $\alpha, s_2 \in \mathbb{R}$, $1 < s, q < \infty$ and $-\frac{n}{s} < \alpha \leq 0$. We suppose that $\frac{n}{s} + \alpha = \frac{n}{q} - s_2$. Let*

$$1 < q < s < \infty,$$

or

$$1 < s \leq q < \infty \text{ and } \alpha < \frac{n}{q} - \frac{n}{s}.$$

Then

$$B_{q, q}^{s_2} \hookrightarrow \dot{K}_s^{\alpha, q}.$$

By the same examples of [2], the assumptions $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$ and $\alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}$ are necessary. Indeed, let $\eta \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\text{supp } \mathcal{F}\eta \subset \{\xi \in \mathbb{R}^n : 3/4 < |\xi| < 1\}$. For $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$ we put $f_N(x) = \eta(2^N x)$. First we have $\eta \in \dot{K}_s^{\alpha_1, p} \cap \dot{K}_q^{\alpha_2, r} \cap \dot{K}_q^{\alpha_2, p}$. Due to the support properties of the function η we have for any $j \in \mathbb{N}_0$

$$\mathcal{F}^{-1} \phi_j * f_N = \begin{cases} f_N, & j = N \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}
\|f_N\|_{\dot{K}_s^{\alpha_1,p} B_\beta^{s_1}} &= 2^{s_1 N} \|f_N\|_{\dot{K}_s^{\alpha_1,p}} \\
&= 2^{s_1 N} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p} \|f_N \chi_k\|_s^p \right)^{1/p} \\
&= 2^{(s_1-n/s)N} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p} \|\eta \chi_{k+N}\|_s^p \right)^{1/p} \\
&= 2^{(s_1-\alpha_1-n/s)N} \|\eta\|_{\dot{K}_s^{\alpha_1,p}}.
\end{aligned}$$

The same arguments give

$$\|f_N\|_{\dot{K}_s^{\alpha_1,p} F_\theta^{s_1}} = 2^{(s_1-\alpha_1-n/s)N} \|\eta\|_{\dot{K}_s^{\alpha_1,p}}, \quad \|f_N\|_{\dot{K}_q^{\alpha_2,r} F_\theta^{s_2}} = 2^{(s_2-\alpha_2-n/q)N} \|\eta\|_{\dot{K}_q^{\alpha_2,r}},$$

and

$$\|f_N\|_{\dot{K}_q^{\alpha_2,p} B_p^{s_2}} = 2^{(s_2-\alpha_2-n/q)N} \|\eta\|_{\dot{K}_q^{\alpha_2,p}}.$$

If the embeddings (16) and (22) hold then for any $N \in \mathbb{N}$

$$2^{(s_1-s_2-\alpha_1+\alpha_2-n/s+n/q)N} \leq c.$$

Thus, we conclude that $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$ must necessarily hold by letting $N \rightarrow +\infty$.

Let now $\omega \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\text{supp } \mathcal{F}\omega \subset \{\xi \in \mathbb{R}^n : |\xi| < 1\}$. For $x \in \mathbb{R}^n$ and $N \in \mathbb{Z} \setminus \mathbb{N}$ we put $f_N(x) = \omega(2^N x)$. We have $\omega \in \dot{K}_s^{\alpha_1,p} \cap \dot{K}_q^{\alpha_2,r} \cap \dot{K}_q^{\alpha_2,p}$. It easy to see that

$$\mathcal{F}^{-1} \phi_j * f_N = \begin{cases} f_N, & j = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\|f_N\|_{\dot{K}_s^{\alpha_1,p} B_\beta^{s_1}} = \|f_N\|_{\dot{K}_s^{\alpha_1,p}} = 2^{-(\alpha_1+n/s)N} \|\omega\|_{\dot{K}_s^{\alpha_1,p}}.$$

The same arguments give

$$\begin{aligned}
\|f_N\|_{\dot{K}_s^{\alpha_1,p} F_\theta^{s_1}} &= 2^{-(\alpha_1+n/s)N} \|\omega\|_{\dot{K}_s^{\alpha_1,p}} \\
\|f_N\|_{\dot{K}_q^{\alpha_2,r} F_\theta^{s_2}} &= 2^{-(\alpha_2+n/q)N} \|\omega\|_{\dot{K}_q^{\alpha_2,r}}
\end{aligned}$$

and

$$\|f_N\|_{\dot{K}_q^{\alpha_2,p} B_p^{s_2}} = 2^{-(\alpha_2+n/q)N} \|\omega\|_{\dot{K}_q^{\alpha_2,p}}.$$

If the embeddings (16) and (22) hold then for any $N \in \mathbb{Z} \setminus \mathbb{N}$

$$2^{-(\alpha_1-\alpha_2+n/s-n/q)N} \leq c.$$

Thus, we conclude that $\alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}$ must necessarily hold by letting $N \rightarrow -\infty$.

5 Applications

In this section, we give a simple application of Theorems 3 and 6. Let w denote a positive, locally integrable function and $0 < p < \infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f\|_{L^p(\mathbb{R}^n, w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $\varrho \in [1, \infty)$ we denote by \mathcal{A}_ϱ the Muckenhoupt class of weights, and $\mathcal{A}_\infty = \cup_{\varrho \geq 1} \mathcal{A}_\varrho$. We refer to [6] for the general properties of these classes. Let $w \in \mathcal{A}_\infty$, $s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel-Lizorkin spaces $F_{p,q}^s(w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p,\beta}^s(w)} = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1}\varphi_j * f|^\beta \right)^{1/\beta} \right\|_{L^p(\mathbb{R}^n, w)}$$

is finite. In the limiting case $q = \infty$ the usual modification is required. Also we define weighted Besov spaces $B_{p,q}^s(w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,\beta}^s(w)} = \left(\sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1}\varphi_j * f\|_{L^p(\mathbb{R}^n, w)}^\beta \right)^{1/\beta}$$

is finite. In the limiting case $q = \infty$ the usual modification is required. The spaces $F_{p,\beta}^s(w)$ and $B_{p,\beta}^s(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces, Banach spaces for $p, q \geq 1$, moreover for $w \equiv 1 \in \mathcal{A}_\infty$ we obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces. Let w_γ be a power weight, i.e., $w_\gamma(x) = |x|^\gamma$ with $\gamma > -n$. Then in view of the fact that $L^p = \dot{K}_p^{0,p}$, we have

$$\|f\|_{A_{p,\beta}^s(w_\gamma)} \approx \|f\|_{\dot{K}_p^{\frac{\gamma}{p}, p} A_\beta^s}.$$

Applying Theorems 3 and 6 in some particular cases yields the following embeddings.

Corollary 1 *Let $s_1, s_2 \in \mathbb{R}$, $0 < q < s < \infty$, $0 < \beta \leq \infty$ and $w_{\gamma_1}(x) = |x|^{\gamma_1}$, $w_{\gamma_2}(x) = |x|^{\gamma_2}$, with $\gamma_1 > -n$ and $\gamma_2 > -n$. We suppose that*

$$s_1 - \frac{n + \gamma_1}{s} = s_2 - \frac{n + \gamma_2}{q}$$

and

$$\gamma_2/q \geq \gamma_1/s.$$

Then

$$F_{q,\beta}^{s_2}(w_{\gamma_2}) \hookrightarrow B_{s,q}^{s_1}(w_{\gamma_1}) \quad \text{and} \quad B_{q,s}^{s_2}(w_{\gamma_2}) \hookrightarrow F_{s,\beta}^{s_1}(w_{\gamma_1}).$$

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